Fractal and chaotic behavior of circular cellular automata

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A new type of circular cellular automata (CCA) has been introduced. The evolutions of the CCA obtained by the clockwise, anticlockwise, and scanning line-by-line site sequence in the successively growing rings divided from a square lattice are studied. The evolution seems to form a twisty fishnet when the CCA are grown by the first two sequences. Sierpinski triangle gasket or the modulated ones are formed in the fourth quadrant of the CCA grown by the line scanning sequence. Fractal analysis is used to characterize the relationships between the pattern formed and the initial position of the growing ring and it is found that the pattern is very sensitive to the initial growth condition, showing the chaotic behavior.

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I. INTRODUCTION

Cellular automata (CA) are nonlinear systems in which space, time, and states are discrete. Their enormous computation speed and complex dynamical behavior make them widely applicable to modeling natural phenomena. Generally, CA may be described in terms of two concepts: configuration and transition rule. Various rules produce different evolution patterns. Wolfram [1] studied all the evolutions of the one dimensional linear CA (LCA) and found that they can be reduced to four classes: homogeneous state (class I), separated simple stable or periodic structures (class II), chaotic pattern (class III) and complex localized structures (sometimes long-lived) (class IV). Sales, Martins, and Moreira [2] found that the Hurst exponent of the spatial and temporal configurations of LCA can detect the existence of long-range correlation in the evolution patterns of class IV and a subclass of class II. The relationship between LCA and fractals has been given by Willson [3,4] and that between LCA and multifractal by Takahashi [5]. CA can also be two dimensional such as the famous game of life [6]. Recently, CA have been widely used for modeling crystal growth, earthquake [7], excitable media [8], granular flow [9], diffusion [10], traffic flow [11,12], and other problems of self organization.

Most of the LCA can be used to simulate one-dimensional growth and flow, but the actual growth and flow usually happen in two- or three-dimensional space. Here, we present a model of a circular cellular automata (CCA) grown on two-dimensional lattice with a simple transition rule. This CCA is sensitive to the sequence for selecting sites in the successively growing rings divided from a square lattice and their initial positions. We have suggested a random successive growth model with a suitable crowded condition for simulating the fractal crystallization in an amorphous film and colony growth patterns [13,14]. Now the CCA is a deterministic model, which can generate fractals, including the famous Sierpinski triangle gasket. The fractal dimension can be used to characterize the patterns produced by the CCA and to show their chaotic behavior.

II. MODEL OF THE CIRCULAR AUTOMATON

It is well known that the original one-dimensional linear cellular automata were grown line by line in one direction and the sites in a line are updated in parallel. But CCA is a two-dimensional model, so that the space is divided by continuously growing rings with the same width. All the sites in a ring are considered to be grown successively. In this CCA model, the growth in each ring is not synchronous and different sequences for selecting sites are applied. The state of a local site depends not only on the states of sites at the last period but also on the states of the nearest sites at current period.

The CCA are constructed as follows: A seed with a value of one (indicated by a dark dot) is placed at the center of a square lattice. All the sites in the ring $(i-1+\delta,\ i+\delta]$ with a width of one ($i=1,2,\ldots,i+\delta$ is the inner radius of the rings and $0\leqslant\delta<1$) are the next sites to be considered to grow or not to grow. A local site is considered to grow (change from 0 to 1) only when just one nearest site is a dark dot among the four nearest sites, and is considered not to grow when no site (too alone) or two sites (too crowded) are dark dots. This condition can be called the intermediate crowded condition.

It is well known that the evolution of LCA is determined by the update rule in parallel, i.e., the influence of neighboring sites in a line is not considered. Contrary to LCA, another important condition of this CCA is the sequence for selecting sites in each ring of $(i-1+\delta,\ i+\delta]$. Three sequences (clockwise, anticlockwise, and scanning line by line from above to below and from left to right in the ring) are considered in this paper. The purpose of this paper is to introduce the CCA model with the sequences for selecting sites, so that the intermediate crowded condition can be included. It is found that complex and interesting patterns can be formed with the change of the simple selecting sequence. In the following text, we call the third sequence as line scanning for convenience.

In this model, there are two variable initial conditions. One is the initial value of δ, and the other is the selecting sequence in the rings. Figure 1 is the sketch of this model with $\delta=0$ grown by the line scanning sequence, where I, II, III, and IV indicate the first, second, third, and fourth quad-

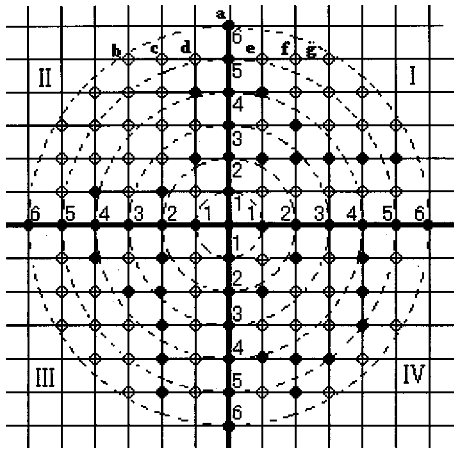


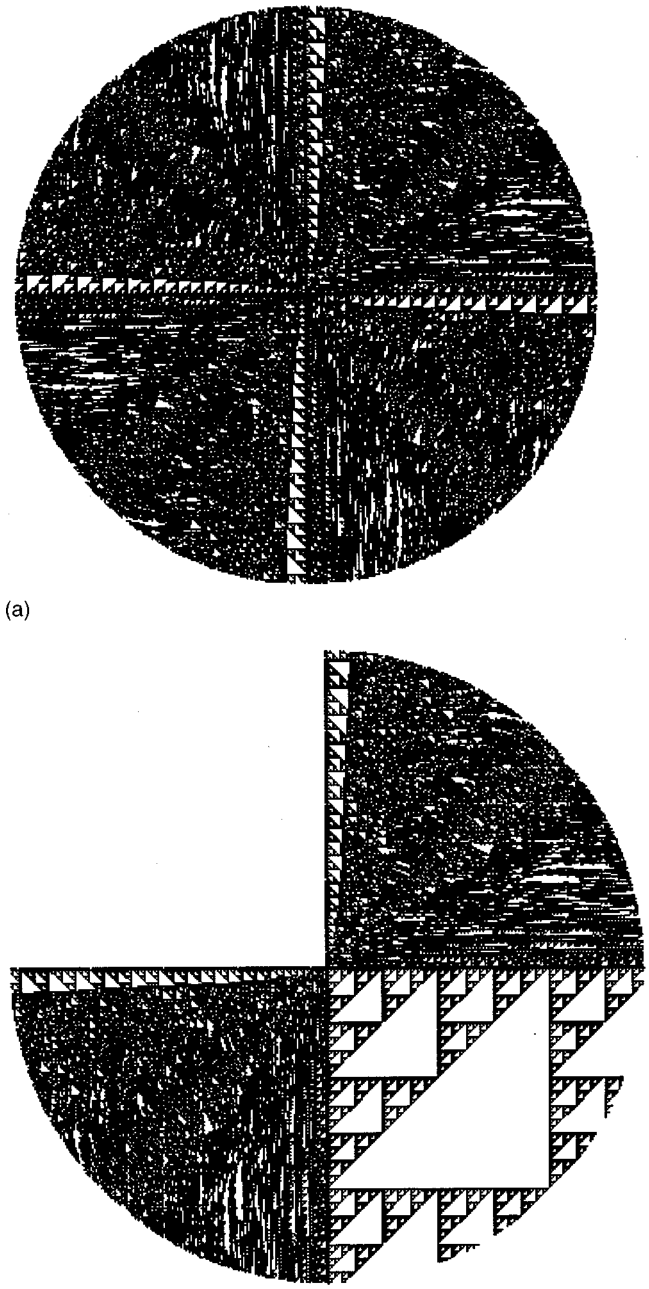
FIG. 1. Sketch of circular cellular automata (CCA) model.

rants. The numbers labeled in the figure are the serial numbers of rings. There are four sites in the first ring of (0, 1], and 8, 16, 20, 32, 32, . . . , sites in next rings ($i=2,3,4,5,\ldots$) when $\delta=0$. The dark dots and the open dots represent those sites grown or not, respectively. *a*, *b*, *c*, *d*, *e*, *f*, *g* in the sixth ring show the site sequence determined by the line scanning sequence. The grown sites in the first quadrant and the third quadrant show mirror symmetry, while those in the second quadrant and the fourth one are not. In the second quadrant, no more sites are grown except the sites on *X* axis and *Y* axis or a part of sites neighboring these axes.

III. RESULTS AND DISCUSSION

Figures 2(a) and 2(b) are the evolution of CCA with $\delta=0$, where $i\leqslant 365$. The grown sequences are clockwise and line scanning in the rings, respectively. Each quadrant of Fig. 2(a) seems to be a twisty fishnet and the figure displays 90° rotation symmetry. The anticlockwise figure shows mirror symmetry to that of the clockwise pattern. As to the Fig. 2(b), although the first quadrant is the same as that of Fig. 2(a) and the third quadrant is the same as that of anticlockwise pattern, most of the sites in the second quadrant cannot grow and interestingly, the pattern in the fourth quadrant is the famous Sierpinski triangle gasket. Figure 2(b) itself has a symmetric mirror at an angle of 45° with the *X* axis.

Figures 3(a), 3(b), 3(c), 3(d), 3(e), and 3(f) are the evolutions of CCA in the 4th quadrant with $\delta=0.04$, 0.07, 0.12, 0.17, 0.20, 0.27 grown by the line scanning sequence in the rings. The evolution changes due to the sites and their number in each ring are different from that of $\delta=0$. The patterns formed in the first quadrant and the third are similar to Fig. 2(b). The patterns in the fourth quadrant are composed of many big or small empty triangles different from the regular Sierpinski gasket and can be called as modulated Sierpinski gasket. They are symmetric with a mirror axis at 45° with the *X* axis. All the second quadrants with various δ are nearly empty. If the sequence for selecting sites in the second quadrant is changed to line scanning from below to above and from left to right, we will obtain the same patterns as that of Fig. 3 instead of the nearly empty quadrant shown in Fig. 2(b). The same patterns can be obtained by the vertical line scanning sequence in the first and third quadrants. So we can get the modulated Siepinski gasket with 90° rotation symmetry with various sequence of selecting sites in the first, second, third, and fourth quadrants.



(a)

(b)

FIG. 2. Evolution of CCA with $\delta=0$. The grown sequences are (a) clockwise (b) line scanning from above to below and from left to right in the rings.

We simulate CCA with the above clockwise and line scanning sequences 100 times with the δ interval of 0.01 at first. All the evolutions are different with each other. Another 500 simulations are done under the sequence of line scanning in the rings when the δ interval decreases to 0.002. The same evolutions are not found in these simulations.

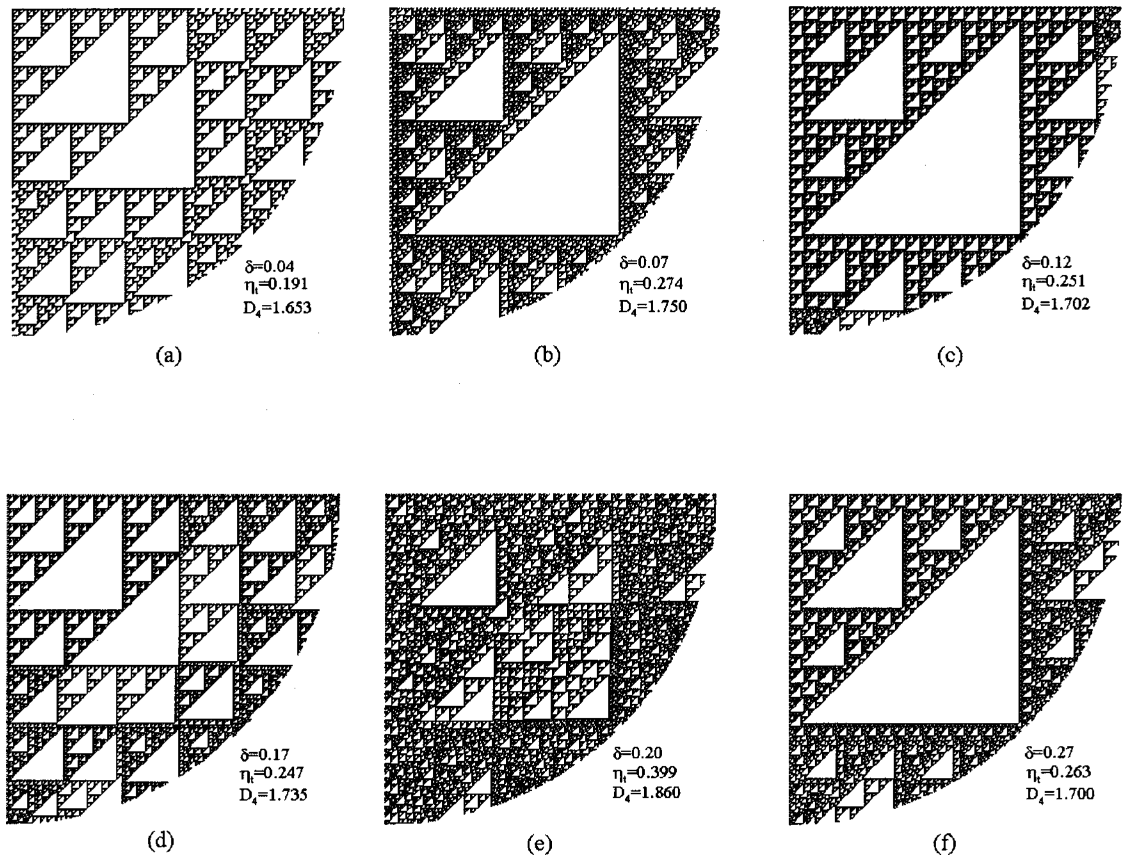


FIG. 3. Evolutions of CCA in the fourth quadrant grown by line scanning from above to below and from left to right in the ring with (a) $\delta=0.04$, (b) $\delta=0.07$, (c) $\delta=0.12$, (d) $\delta=0.17$, (e) $\delta=0.20$, (f) $\delta=0.27$, where η_t is the total occupation percentage and D_4 is the fractal dimension.

Figure 4(a) shows radius (i) dependence of the occupation percentage(η) in a ring in the first quadrant and fourth quadrant with $\delta=0$ in the case of line scanning sequence, where η is defined as the ratio of the number of the occupied sites to that of all the sites of the ring of $(i-1+\delta,\ i+\delta]$ in the quadrant. The occupation percentages in the 1st quadrant are quite close with small random fluctuations. However, the η of the fourth quadrant show strong fluctuations and a regularity can be found from random fluctuations. There are sudden rises of η in the curve of $\eta-i$. The abrupt raising sites are marked with the serial numbers of the ring in the evolution. It can be easily found that the serial numbers increase with a power law of 2^n. The variations of η in the first quadrant and fourth quadrant with i at $\delta=0.2$ are also plotted in Fig. 4(b). The situation of η of the first quadrant has no obvious change, while that of the fourth quadrant is quite different. Although the fluctuations are still random and relatively large, the regularity disappears. This situation is similar with other $\delta>0$ and the variations are sensitive to the initial value of δ.

The total occupation percentages η_t in each quadrant have been examined when δ increases from 0 to 1. η_t has no large variation in the first or third quadrant. It ranges from 0.463 to 0.488. In the fourth quadrant, η_t is generally less than that in the first or third quadrant and sensitive to the initial value of δ. The upper limit of η_t in this quadrant is 0.477, but the lowest limit of η is only 0.133, corresponding to the lowest

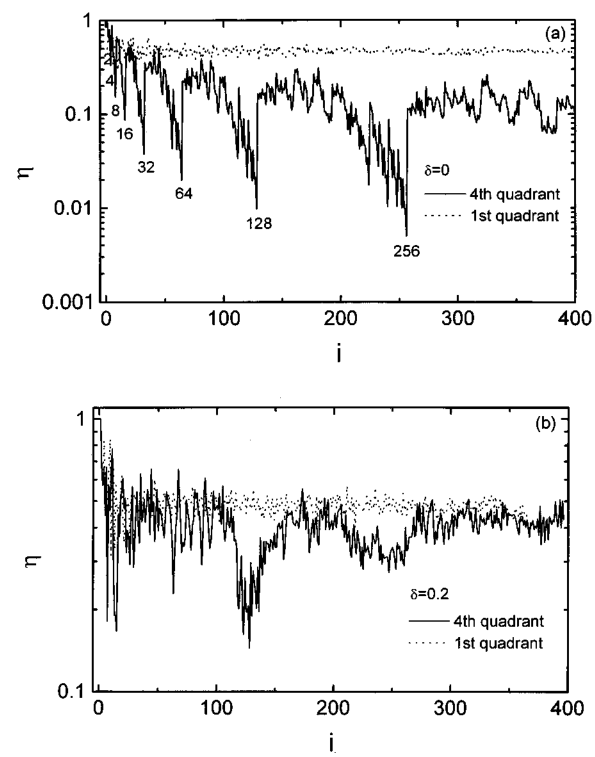


FIG. 4. Radius (i) dependence of η in the first quadrant and fourth quadrant (a) $\delta=0$ (b) $\delta=0.2$.

fractal dimension in the case of formation of Siepinski gasket. According to the growth rule, the cluster formed by this CCA is infinite. But the occupation percentages at every initial value of δ are less than the threshold value of site percolation model in square lattice (0.59). This is due to that in the random percolation model the formation of an infinite cluster is accompanied by the formation of many small isolated clusters, while in the CCA model the infinite paths are formed by the intermediate crowded growth rule, similar to a self-organization process.

As mentioned above, the fourth quadrant of the evolution at $\delta=0$ is a Sierpinski triangle gasket. It is a regular fractal with a fractal dimension of $\ln 3/\ln 2=1.584\,96$. Certain variation happens and modulated Sierpinski triangle gaskets appear in this quadrant when $\delta>0$. Fractal analysis can be used to describe this kind of variation quantitatively.

The fractal dimension is calculated by box-counting method: A 256×256 square is cut out from a quadrant with the start point at the seed site. Boxes with size ε ($\varepsilon\leqslant1$, $\varepsilon=1$ when $x_0=y_0=256$ where x_0 and y_0 are abscissas along X axis and Y axis) are covered on the 256×256 square pattern in the quadrant. The number of boxes (N) which have occupied sites in it is counted. Changing the box size, a series of N and ε can be obtained. The patterns can be viewed as fractals if the $\ln N$-$\ln\varepsilon$ curve is straight and the slope of $\ln N$-$\ln(\varepsilon)$ within the linear range is the fractal dimension D.

The $\ln N$-$\ln(\varepsilon)$ curve of the fourth quadrant at $\delta=0$ (regular Sierpinski triangle gasket) shows ideal linearity at all the studied ε range. The calculated fractal dimension is just equal to $\ln 3/\ln 2$. The linearity of $\ln N$-$\ln(\varepsilon)$ curve at $\delta>0$ is good too (The correlation coefficient of the $\ln N$-$\ln(\varepsilon)$ curve at $\delta>0$ is about 0.998). The fractal dimension D_4 at $\delta>0$ is larger than $\ln 3/\ln 2$ due to the growth pattern at $\delta>0$ is more compact than the regular Sierpinski triangle gasket. Figure 5 shows the dependence of fractal dimension D_4 of the fourth quadrant with δ. In general, D_4 increases with δ, but there are many irregular fluctuations and the fractal dimension D_4 is different with each initial value of δ.

The patterns in the first quadrant can also be analyzed by fractal. The dependence of fractal dimension D_1 on δ is shown in Fig. 5 too. Although the fractal dimension D_1 is concentrated around 1.916, there are many irregular fluctuations in D_1-δ curve. The same values of fractal dimension are not found with different initial value of δ, similar to the situation of the fourth quadrant. This means the CCA are sensitive to the initial value of δ, showing its chaotic behavior.

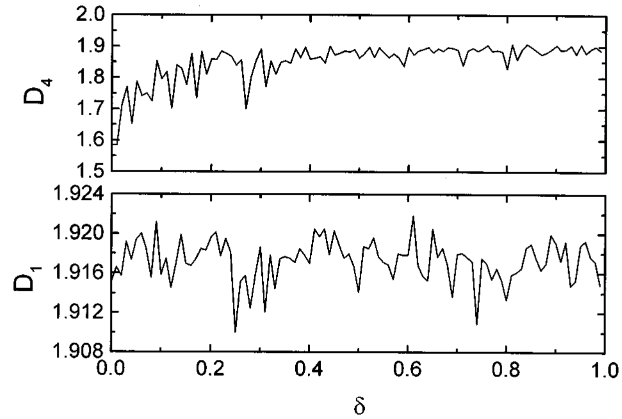


FIG. 5. Dependence of fractal dimension of the patterns in fourth and first quadrants on δ.

The CCA presented above is a simple two-dimensional growth model. Further work is required to develop this model. For example, the initial size of seed and width of rings can be enlarged. This model is deterministic, but it can be changed when a growth probability is added to the intermediate crowded growth condition. Then this model can be used to improve our previous work of the random successive growth model for pattern formation and bacterial-colony growth that includes also the intermediate crowded condition [13,14]. The CCA can also be developed to the three-dimensional space as a spherical CA with various sequences for selecting sites in the successively growing spherical shells.

IV. CONCLUSION

In summary, a new circular CA has been introduced and the fractal analysis has been used to discuss the two-dimensional pattern formation of CCA. The fractal dimensions of the patterns in the first quadrant (twisty fishnet) and fourth quadrant (modulated Sierpinski triangle gasket) are sensitive to the initial condition of the CCA, showing the chaotic behavior.

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